COMPLEX NUMBERS

Intro to iota and Powers of iota

Complex number, represented by the "z", written as, $z = x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$

$i^{4w} = 1$	$i^{4w+1} = i$	$i^{4w+2} = -1$	$i^{4w+3} = -i$

Trick to Remember : +(One Eye) -(One Eye)

Important : $i^{4w} + i^{4w+1} + i^{4w+2} + i^{4w+3} = 0$

Real & Imaginary Parts in $z = x + iy$

$Re(z) = x$	If $Re(z) = 0$, Purely Imaginary Number
$Im(z) = y$	If $Im(z) = 0$, Purely Real Number

$0 + i0$ is purely Real & Purely imaginary but not imaginary

Algebra of Complex numbers

Addition	$(a+ib)+(c+id) = (a+c)+i(b+d)$
Subtraction	$(a+ib)-(c+id) = (a-c)+i(b-d)$
Multiplication	$(a+ib)(c+id) = (ac-bd)+i(ad+bc)$

- if, $z_1 = z_2$ a, $Re(z_1) = Re(z_2)$ & $Im(z_1) = Im(z_2)$
- $z_1^2 + z_2^2 = 0$ doesn't imply $z_1 = z_2 = 0$
- Inequality of imaginary numbers is not discussed
  - if $(a+ib) > (c+id)$, then $b=d=0$

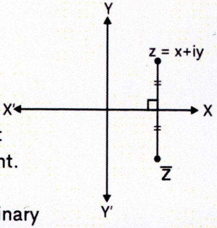
Multiplicative inverse	$\frac{1}{a+ib} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$

Terms associated with Complex Numbers

Conjugate (Change sign of imaginary part)

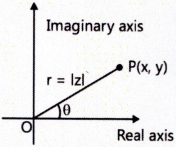
If $z = x + iy$, then $\bar{z} = x - iy$

- Properties followed
  - $z + \bar{z} = 2\text{Re}(z)$
  - $z - \bar{z} = 2i\ \text{Im}(z)$
  - $z.\bar{z} = x^2 + y^2$
  - If z lies in 1st quadrant then $\bar{z}$ lies in 4th quadrant and $-\bar{z}$ in the 2nd quadrant.
  - If $z + \bar{z} = 0 \Rightarrow$ Purely Real
  - If $z - \bar{z} = 0 \Rightarrow$ Purely Imaginary



Modulus (Distance of complex no. from origin)

- If $z = x + iy$, then $|Z| = \sqrt{x^2 + y^2}$
- Complex number satisfying $|z| = r$ lie on a circle having radius = r
- Angle θ made anticlockwise with x-axis is **Argument**. The unique value of $\theta \in (-\pi, \pi]$ is called **Amplitude** or **Principle value of argument .**
  - **Least Positive Argument** = $\theta = \tan^{-1}\left|\frac{y}{x}\right|$



Properties of Conjugate, Modulus & Argument

Conjugate

$$\overline{(\bar{z})} = z$$

$$\overline{(z_1 \times z_2)} = \overline{z_1} \times \overline{z_2}$$

$$\overline{(z_1 \pm z_2)} = \overline{z_1} \pm \overline{z_2}$$

$$\overline{(z^n)} = (\bar{z})^n$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \left(\frac{\overline{z_1}}{\overline{z_2}}\right)$$

Modulus

$$|z|^n = |z^n|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$z\bar{z} = x^2 + y^2 = |z|^2$$

$$|z_1 + z_2| \le |z_1| + |z_2|$$

$$|z_1 - z_2| \ge |z_1| - |z_2|$$

$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

$$|z| = 0 \Rightarrow z = 0$$

$$|\bar{z}| = |z| = |-z| = |-\bar{z}|$$

$$|z_1 z_2| = |z_1||z_2|$$

$$\text{if } |z| = a, \text{ then } z = \frac{a^2}{\bar{z}}$$

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

$$|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\mathrm{Re}(z_1\overline{z_2})$$

Argument (Amplitude)

$$\text{amp}(z_1 z_2) = \text{amp}(z_1) + \text{amp}(z_2) + 2k\pi, k \in I$$

$$\text{amp}\left(\frac{z_1}{z_2}\right) = \text{amp}(z_1) - \text{amp}(z_2) + 2k\pi, k \in I$$

$$\text{amp}(z^n) = n.\text{amp}(z) + 2k\pi$$

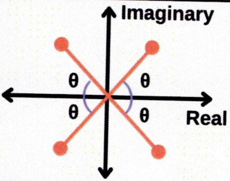
$$\text{amp}(\text{postive real no.}) = 0$$

$$\text{amp}(\text{negative real no.}) = \pi$$

$$\text{amp}(z - \bar{z}) = \pm {}^{\pi}/_{2}$$

$$\text{amp}(\bar{z}) = -\text{amp}(z) = \text{amp}(1/z)$$

Amplitude of a complex in different Quadrants



Quadrant 1: $\arg(z) = \theta$

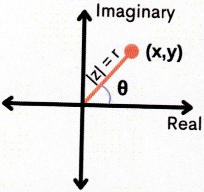
Quadrant 2: $\arg(z) = \pi - \theta$

Quadrant 3: $\arg(z) = \theta - \pi$

Quadrant 4: $\arg(z) = -\theta$

Representation of Complex numbers

- **Cartesian Form**
  - $z = x + iy$
- **Polar Form**
  - $z = r(\cos\theta + i\sin\theta)$
- **Euler Form**
  - $z = re^{i\theta}$



Use Euler form where powers are very high

$$z = re^{i\theta} = \cos\theta + i\sin\theta$$

De-Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Cube Roots of Unity, $z^3 = 1$

- Cube Roots of Unity are :
  - $1, \underbrace{-\frac{1}{2} + i\frac{\sqrt{3}}{2}}_{\omega}, \underbrace{-\frac{1}{2} - i\frac{\sqrt{3}}{2}}_{\omega^2 \text{ or } \overline{\omega}}$

- $1 + \omega + \omega^2 = 0$ & $\omega^3 = 1$
- Three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.

- Roots for $z^2 + z + 1 = 0$ is ω, ω^2
- Roots for $z^2 - z + 1 = 0$ is $-\omega$, $-\omega^2$

REMEMBER!

- Other Factorisations to be remembered

$$a^3 - b^3 = (a-b)(a-\omega b)(a-\omega^2 b)$$

$$x^2 + x + 1 = (x-\omega)(x-\omega^2)$$

$$a^3 + b^3 = (a+b)(a+\omega b)(a+\omega^2 b)$$

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c)$$

- In Polar form, Cube Roots of Unity are

$$\cos 0 + i\sin 0;\ \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3};\ \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$

SUMMATION OF SERIES USING COMPLEX NUMBER

$$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots\ldots + \cos n\theta = \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cos\left(\frac{n+1}{2}\right)\theta$$

$$\sin\theta + \sin 2\theta + \sin 3\theta + \ldots\ldots + \sin n\theta = \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \sin\left(\frac{n+1}{2}\right)\theta$$